Solution to Take home examination
in Basic stochastic processes 2009

Day assigned: November 23, 10:00 am
Due date: November 24, 10:00 am

- The take home examination is strictly individual. Submissions that bear signs of being collective efforts will be disregarded.
- Students are supposed to give a precise description of the model used to solve the problem and rigorous explanations to the solution.
- Correct answers without explanations will be disregarded.

Problem 1. Let $\{N(t),\ t \geq 0\}$ be the Poisson process with rate λ.

(a) Fix the point $t > 0$ and denote by X the time distance between the last arrival of the process before t to the first arrival after t, when $N(t) > 0$, and the distance from zero to the first arrival, when $N(t) = 0$. What is true: $E[X] < \frac{1}{\lambda}$; $E[X] = \frac{1}{\lambda}$; $E[X] > \frac{1}{\lambda}$?
Explain your answer. 2p

(b) For $t \geq 0$, define the random process

$$Y(t) = (-1)^{N(t)}.$$

Compute the distribution and the expected value of $Y(t)$ for $t > 0$. 3p

(c) Are the increments of $Y(t)$ independent? Are they stationary? Prove your answers. 2p

Solution.

(a)

$$\begin{aligned} E[X] &= E[S_{N(t)+1} - t] + E[t - S_{N(t)}] \\ &= E[\gamma_t] + E[\delta_t] = \frac{1}{\lambda} + E[\delta_t] > \frac{1}{\lambda}, \end{aligned}$$

since

$$E[\delta_t] \geq t\, P\{\delta_t = t\} = t\, e^{-\lambda t} > 0.$$

(b) The possible values of $Y(t)$ are ± 1.

$$P\{Y(t) = 1\} = P\{N(t) \text{ is even }\} = \sum_{n-even} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = p_e(t)$$

$$P\{Y(t) = -1\} = P\{N(t) \text{ is odd }\} = \sum_{n-odd} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = p_o(t)$$

We have

$$p_e(t) + p_o(t) = 1,$$

$$p_e(t) - p_o(t) = \sum_{n-even} \frac{(\lambda t)^n}{n!} e^{-\lambda t} - \sum_{n-odd} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} e^{-\lambda t} = e^{-2\lambda t}.$$

Thus

$$p_e(t) = \frac{1 + e^{-2\lambda t}}{2}, \quad p_o(t) = \frac{1 - e^{-2\lambda t}}{2}$$

and

$$E[Y(t)] = 1 \cdot p_e(t) - 1 \cdot p_o(t) = e^{-2\lambda t}.$$

(c) Possible values of the increments are -2, 0, 2. For $t > 0$, consider the increments

$$Y(2t) - Y(t), \quad Y(t) - Y(0).$$

We have

$$P\{Y(2t) - Y(t) = 2,\ Y(t) - Y(0) = 0\}$$
$$= P\{Y(2t) = 1,\ Y(t) = -1,\ Y(t) = 1\} = 0$$

while

$$P\{Y(2t) - Y(t) = 2\} = P\{Y(2t) = 1,\ Y(t) = -1\}$$
$$= P\{Y(2t) = 1 | Y(t) = -1\}P\{Y(t) = -1\}$$
$$= p_o^2(t) > 0$$

and

$$P\{Y(t) - Y(0) = 0\} = P\{Y(t) = 1\} = p_e(t) > 0$$

Hence the above increments are not independent. Neither are they stationary, since they both are increments on intervals of length t, but have different distributions: $Y(2t) - Y(t)$ takes on values -2, 0, and 2, while $Y(t) - Y(0)$ takes on values -2 and 0.

Problem 2. Passengers arrive at a bus stop according to a Poisson process with rate λ. Buses depart from the stop according to a renewal process with interdeparture times $A_1, A_2, \ldots$. We assume that the buses have ample capacity so that all waiting passengers get in the bus that departs. Use a proper renewal-reward process to prove that the long-run average waiting time per passenger equals $\dfrac{E[A_1^2]}{2E[A_1]}$. Can you give a heuristic explanation of why the answer for the average waiting time is the same as the average residual life in the renewal process? 3p

Solution. Put the time origo at a moment when a bus departs from the stop, and let N_k be the number of passengers arriving at the stop between the $(k-1)$th and the k-th departure. Define W_n to be the waiting time of the nth passenger. The process $\{W_n,\ n \geq 1\}$ is regenerative with cycles $N_1,\ N_2,\ \ldots$. Let W be the total waiting time of the passengers in one cycle. The long-run average waiting time per passenger is given by

$$\frac{E[W]}{E[N_1]}$$

We have

$$E[N_1|A_1 = a] = \lambda a$$
$$E[N_1|A_1] = \lambda A_1$$
$$E[N_1] = \lambda E[A_1]$$

and (Example 1.1.5)

$$E[W|A_1 = a] = \frac{\lambda a^2}{2}$$
$$E[W|A_1] = \frac{\lambda A_1^2}{2}$$
$$E[W] = \frac{\lambda E[A_1^2]}{2}$$

The long-run average waiting time per person is then

$$\frac{\lambda E[A_1^2]/2}{\lambda E[A_1]} = \frac{E[A_1^2]}{2E[A_1]}$$

The results is the same as the long-run average waiting time for a bus to come of a passenger that comes at a random time point. The two long-run average waiting times are the same due to the property PASTA of the Poisson process.

Problem 3. A system consists of N identical machines maintained by a single repairman. The machines operate independently of each other and each machine has an exponential life time with mean value $1/\mu$. The system fails when the number of failed machines has reached a given critical level R, where $1 \leq R < N$. Then all failed machines are repaired simultaneously.

(a) Compute the average time until the system fails. 3p

(b) The system is started up immediately after the failed machines have been repaired. Assume that any repair takes a negligible time and a repaired machine is again as good as new. The cost of the simultaneous repair of R machines is $K + cR$, where K and c are positive constants. Also there is an idle-time cost of $\alpha > 0$ per time unit for each failed machine. In the long run, what is the average total cost per time unit? 2p

Solution

(a) Let X_i be the life-time of machine i, $\quad i = 1, 2, \ldots N$. Since these random variables are independent, the time until the first failure of a machine

$$Y_1 = min\{X_1,\ X_2,\ \ldots, X_N\}$$

is an exponential random variable with mean $\frac{1}{N\mu}$. Using the memoryless property of the exponential distribution we see that the average time from the first failure to the second one is $\frac{1}{(N-1)\mu}$, and so on. Thus the average length of the time needed for R failures (or the time before the system fails) is

$$\frac{1}{N\mu} + \frac{1}{(N-1)\mu} + \ \ldots + \frac{1}{(N-R+1)\mu}$$

(b) The continuous-time stochastic process describing the number of operating machines at time t regenerates any time the system fails and has average cycle length as obtained in (a).

The average cost incurred in one cycle is

$$\frac{\alpha}{(N-1)\mu} + \frac{2\alpha}{(N-2)\mu} + \ldots\ldots \frac{(R-1)\alpha}{(N-R+1)\mu} + K + cR.$$

The long-run average cost per time unit is given by the ratio of the above expression and the one in (a).